2022-2023 Senior Mathematical Olympiad

Round Two Examination (Grades 9,10 and 11) - 11:00am

SECTION A

For each question, determine the letter corresponding to the correct or best response; along with the question number, indicate this letter by shading it on the answer sheet

1. What is the smallest four-digit positive integer which has four different digits?

   (A) 1032   (B) 2012   (C) 1021   (D) 1234   (E) 1023

2. In the following expression each $\square$ is to be replaced with either $+$ or $-$ in such a way that the result of the calculation is 100.

$$123 \,\square\, 45 \,\square\, 67 \,\square\, 89$$

   Let p be the number of $+$ signs used and m the number of $-$ signs used. What is the value of $p - m$?

   (A) -3 (B) -1 (C) 0   (D) 1   (E) 3

3. How many two-digit numbers have remainder 1 when divided by 3 and remainder 2 when divided by 4?

   (A) 8   (B) 7   (C) 6   (D) 5   (E) 4

4. Which one of the following is equal to

$$\sqrt{9^{16x^2}}$$

   for all values of x?

   (A) 3^{4x} (B) 3^{4x^2} (C) 3^{8x^2} (D) 9^{4x} (E) 9^{8x^2}

5. After playing 500 games, Sarah's success rate at Solitaire is 49%. If Sarah wins every game from now on, how many more games does she need to play in order that her success rate increases to 50%?

   (A) 1   (B) 2   (C) 5   (D) 10   (E) 50

6. $PQRS$ is a quadrilateral inscribed in a circle of which PR is a diameter. The lengths of PQ, QR and RS are $60, 25$ and 52 respectively.
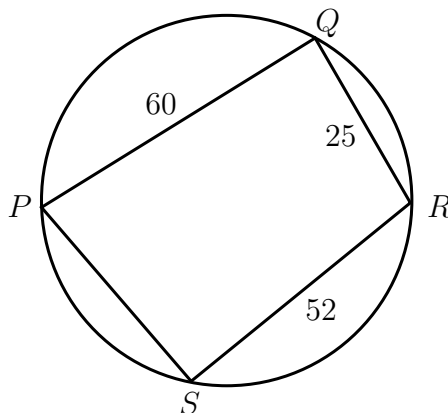
   What is the length of SP?

   (A) $21\frac{2}{3}$ (B) $28\frac{11}{13}$ (C) 33   (D) 36   (E) 39

7. Peter wrote a list of all the prime numbers that could be produced by changing one digit of the number 200. How many of the numbers in Peter's list are prime?

   (A) 0   (B) 1   (C) 2   (D) 3   (E) 4

8. Two externally tangent circles (Circles touching each other) with centers at points A and B have radii of lengths 5 and 3, respectively. A line externally tangent to both circles intersects ray AB at point C. What is the length of BC?

   (A) 4   (B) 4.8   (C) 10.2   (D) 12   (E) 14.4

9. The parallel sides of a trapezium have lengths $2x$ and $2y$ respectively. The diagonals are equal in length, and one diagonal makes an angle θ with the parallel sides as shown.
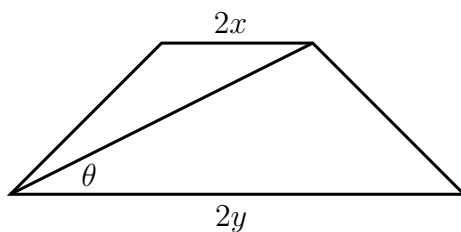
   What is the length of each diagonal?

   (A) $x + y$ (B) $\dfrac{x + y}{\sin \theta}$ (C) $(x + y)\cos \theta$ (D) $(x + y)\tan \theta$ (E) $\dfrac{x + y}{\cos \theta}$

10. The interior angles of a triangle are

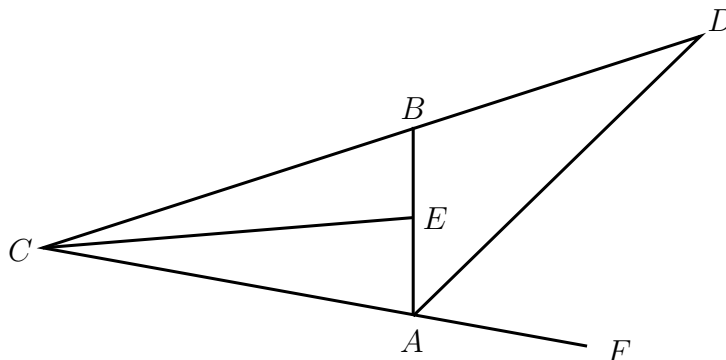$$(5x + 3y)^\circ, (3x + 20)^\circ \text{ and } (10y + 30)^\circ$$

where x and y are positive integers. What is the value of $x + y$?

(A) 15   (B) 14   (C) 13   (D) 12   (E) 11

SECTION B

For each question, provide a complete solution by showing all your workings.

1. The diagram shows triangle ABC, in which $\angle ABC = 72°$ and $\angle CAB = 84°$. The point E lies on AB so that EC bisects $\angle BCA$. The point F lies on CA extended. The point D lies on CB extended so that DA bisects $\angle BAF$.

Prove that $AD = CE$.

2. Let $P(n)$ and $S(n)$ denote the product and the sum, respectively, of the digits of the integer n. For example,

$$P(23) = 6 \text{ and } S(23) = 5.$$

Suppose N is a two-digit number such that $N = P(N) + S(N)$. Determine the units digit of N?

3. The letters a, b, c, d, e and f and represent single digits and each letter represents a different digit. They satisfy the following equations:

$$a + b = d, \quad b + c = e \quad \text{and} \quad d + e = f.$$

One solution for the ordered set (a, b, c, d, e, f) is $(2, 1, 4, 3, 5, 8)$. Find all the other solutions.

4. Two overlapping triangles POR and QOT are such that points P, Q, R and T lie on the arc of a semicircle of centre O and diameter PQ, as shown in the diagram.
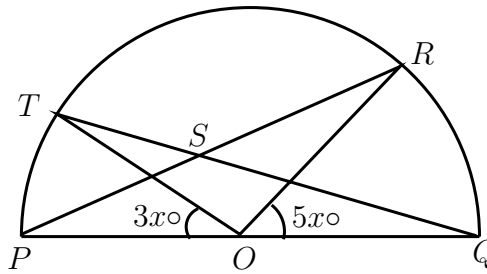
Lines QT and PR intersect at the point S. Angle TOP is $3x°$ and angle ROQ is $5x°$.

In terms of x, what is the measure of $\angle RSQ$?

5. In a sequence, every term after the second is equal to the sum of the previous two terms. Also, every term is a positive integer. The eighth term in the sequence is 400. What is the maximum value of the third term in the sequence.